

Mermin pentagrams arising from Veldkamp lines for three qubits

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Abstract: We study the geometry of the space of Mermin pentagrams, objects that are used to rule out the existence of noncontextual hidden variable theories as alternatives to quantum theory. It is shown that this space of 12 096 possible pentagrams is organized into 1008 families, with each family containing a "double-six" of pentagrams. The 1008 families are connected to a special set of Veldkamp lines in the Veldkamp space for three-qubits an object well-known to finite geometers but has only been introduced to physics recently. Due to the transitive action of the symplectic group on this set of Veldkamp lines it is enough to study only one "canonical" double-six configuration of pentagrams. We prove that the geometry of this double-six configuration is encapsulated in the weight diagram of the 20 dimensional irreducible representation of the group $SU(6)$. As an interesting by-product of our approach we show that Mermin pentagrams and a class of Mermin squares labelled by three-qubit Pauli operators are inherently related. We conjecture that by studying the representation theoretic content of other Veldkamp lines of the Veldkamp space for N -qubits makes it possible to find new contextual configurations in a systematic manner.

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1 Introduction

In the 1990s there was a flurry of activity on revisiting the Kochen-Specker[1] and Bell theorems[2]. Peres[3], Mermin[5, 6] and Greenberger, Horne, Zeilinger[7] have given elegant proofs of these theorems using systems of two, three and four qubits. The remarkable feature appearing in these works was that they were able to rule out certain classes of hidden variable theories without the use of probabilities. Since the advent of quantum information theory these ideas have been again under an intense scrutiny[8, 9, 10, 11, 12, 13, 14, 16]. Apart from obtaining new proofs for these theorems[8, 9] some of these works also attempted to relate the geometric configurations underlying these proofs to finite geometric structures well-known to mathematicians[10, 11, 12, 13], or even to the structure of entropy formulas of some black hole solutions in string theory[14, 15, 16]. The basic idea of these considerations was to label the points of these configurations with the observables of some *finite* dimensional multipartite system in a manner compatible with a set of rules encapsulating the commutation properties and some identities for the observables. The net result of this process was an incidence structure which encapsulated the inherent noncommutativity underlying our multipartite quantum system.

For N -qubit systems an approach of that kind was initiated in [17] with the incidence structure arising called $\mathcal{W}_{2N-1}(2)$ the symplectic polar space of rank N order two[18]. This space is equipped with a symplectic form with values in the two element field \mathbb{Z}_2 , corresponding to the two possibilities for the observables are either commuting or not commuting. The

symplectic group $\text{Sp}(2N, \mathbb{Z}_2) \equiv \text{Sp}(2N, 2)$ is the one leaving invariant the symplectic form (null-polarity) of $\mathcal{W}_{2N-1}(2)$. The main application of this space is within the field of quantum information and it is related to quantum error correcting codes[19]. The construction of such codes is naturally facilitated within the so called stabilizer formalism[19, 20, 21]. Here it is recognized that the basic properties of error correcting codes are related again to the fact that two operators in the Pauli group are either commuting or anticommuting.

Later it has also been realized that certain subconfigurations of $\mathcal{W}_{2N-1}(2)$ called geometric hyperplanes[22] are also worth studying. For example in the case of $\mathcal{W}_3(2)$ one particular class of its geometric hyperplanes features the 10 possible Mermin squares[5, 6] one can construct from two-qubit Pauli operators. For mathematicians it is well-known that sometimes the set of geometric hyperplanes can be organized to Veldkamp lines which are in turn organized into a new incidence structure the so-called Veldkamp space[23, 22]. In this spirit for the simplest nontrivial case of $\mathcal{W}_3(2)$ the structure of the Veldkamp space has been thoroughly investigated, the physical meaning of the geometric hyperplanes clarified, and pictorially illustrated[24]. Since for an arbitrary number of qubits the diagrammatic approach of [24] is not feasible, a later study[25] has revealed the structure of the Veldkamp space of $\mathcal{W}_{2N-1}(2)$ in a purely algebraic fashion.

The $N = 3$ (three-qubits) case is particularly interesting. In this case the symplectic group $\text{Sp}(6, 2)$ associated with $\mathcal{W}_5(2)$ is isomorphic[26] to $W(E_7)/\mathbb{Z}_2$ with $W(E_7)$ the Weyl group of the exceptional group E_7 . This group is the automorphism group of a point-line incidence geometry \mathcal{G}_3 where the 63 points are the points of $\mathcal{W}_5(2)$ and the 315 lines are the totally isotropic ones of $\mathcal{W}_5(2)$ with respect to the symplectic form[25]. A nice way of understanding this is to note that there is a bijection[28] between the 63 points of $\mathcal{W}_5(2)$ and the 63 pairs of roots of E_7 . One particular type of geometric hyperplanes of \mathcal{G}_3 is featuring 27 points and 45 lines and having the incidence geometry of a generalized quadrangle[29] $\text{GQ}(2, 4)$ with the automorphism group $W(E_6)$. In [14] it has been shown how these structures encode information on the structures of the $E_{7(7)}$ and $E_{6(6)}$ symmetric black hole entropy formulas arising in certain models of string theoretic compactifications. It has been also observed[15, 16] that certain truncations of these models result in truncations of the corresponding entropy formulas, which in turn correspond to truncations to further geometric hyperplanes. For example the 27 points of $\text{GQ}(2, 4)$ can be partitioned into three sets of Mermin *squares* with 9 points each. This partitioning corresponds to the reduction of the 27 dimensional irreducible representation of $E_{6(6)}$ to a substructure arising from three copies of 3 dimensional irreps of three $\text{SL}(3, \mathbb{R})$ s. The configuration related to this truncation has an interesting physical interpretation in terms of wrapped membrane configurations and is known in the literature as the bipartite entanglement of three qutrits[30, 16]. These results hint at a deep connection between the structure of geometric hyperplanes (*Veldkamp points*), contextual configurations[31] and the structure of irreducible representations of certain types of Lie-groups.

In this paper we reveal a connection between the set of *Mermin pentagrams* and a particular class of *Veldkamp lines* for three qubits. The number of Mermin pentagrams is 12096 (see [10]), and these organize into 1008 families of 12 pentagrams[32]. We call these families *double sixes of pentagrams*. We would like to understand the geometry of these spaces. We show that each family is inherently connected to the structure of the weight diagram for the 20 dimensional irrep of $\text{SU}(6)$. It turns out that there are 1008 such families that can be mapped bijectively to a subclass of Veldkamp lines of the Veldkamp space for three-qubits. Since there is a transitive action of the symplectic group $\text{Sp}(6, 2)$ on this class of Veldkamp lines[25] it is enough to study merely one particular family, which we call canonical. As a byproduct of our approach we also establish an interesting connection between a subset of the space of Mermin *squares* formed from three-qubit operators and the families of double-sixes of Mermin *pentagrams*.

The organization of this paper is as follows. In Section 2 we collect the relevant background information on the N -qubit Pauli group related to the structure of the symplectic polar space $\mathcal{W}_{2N-1}(2)$. Here our basic notions like geometric hyperplanes, Lagrange subspaces, Veldkamp lines and Veldkamp spaces for N -qubits are defined. In Section 3 the space of Mermin pentagrams for three-qubit systems is introduced. Here we also introduce the notion of a double-six configuration of 12 pentagrams. In Section 4 we demonstrate the connection between the structure of this double-six configuration of Mermin pentagrams and the structure of the spreads of the generalized quadrangle $\text{GQ}(2, 2)$ also called the *doily*[29, 33]. Armed with this observation, in Section 5 we easily prove that a particular set of 1008 Veldkamp lines with their core sets being 1008 doilies encode all relevant information concerning the space of Mermin pentagrams. In fact we have to study merely one particular Veldkamp line, with a particular doily and a particular double-six. The reason for this is that the symplectic group acts transitively[25] on our particular 1008 element set of Veldkamp lines. Hence by uncovering the structure of merely one canonical arrangement for the aforementioned objects we have uncovered the structure of all such ones. In Section 6 we prove that the geometry of the canonical double-six configuration (hence by transitivity all of such ones) is encapsulated entirely in the weight diagram of the 20 dimensional irreducible representation of the group $\text{SU}(6)$. Section 7 is left for the conclusions and some speculations. Here we notice that a certain 10080 element subset of the space of Mermin squares is related to the 12096 element set of Mermin pentagrams in a simple manner. Here we also conjecture that the structure of Veldkamp lines for N -qubits ($N = 2, 3, 4$) is inherently connected to special irreducible representations of ADE-type Lie algebras. This observation might be a starting point for obtaining new magic configurations in a systematic manner.

2 Mathematical background

We start by recapitulating some basic notions and facts about the building blocks of the set of Mermin pentagrams. The *observables* of a quantum system described by a finite dimensional Hilbert space \mathcal{H} are represented by self-adjoint operators on \mathcal{H} , and if a basis is fixed in \mathcal{H} , each observable is equivalent to a Hermitian matrix. In the case of a two-state quantum system, or *qubit*, a basis for the vector space of the 2×2 Hermitian matrixes is $\{I, X, Z, Y\}$, where I is the 2×2 identity matrix, and

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = iXZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (1)$$

are the three *Pauli spin matrices*. The matrices of a special set of observables of a system of N qubits are the N -fold Kronecker products of I, X, Z and Y . We will refer to these $2^N \times 2^N$ matrixes as *N -qubit Pauli operators*.

The N -qubit Pauli operators generate the *N -qubit Pauli group*

$$P_N = \{sA_1A_2\dots A_N \mid s \in \mu_4, A_i = I, X, Z, Y\} \leq GL(2^N, \mathbb{C}), \quad (2)$$

where $A_1A_2\dots A_N$ is a shorthand notation for $A_1 \otimes A_2 \otimes \dots \otimes A_N$, and μ_4 is the group of the fourth roots of unity:

$$\mu_4 = \{\pm 1, \pm i\} \subseteq \mathbb{C} \setminus \{0\}. \quad (3)$$

The commutator subgroup of P_N , i.e. the derived group $P'_N = [P_N, P_N]$ coincides with the center of P_N ,

$$Z(P_N) = \{sI_N \mid s \in \mu_4\}, \quad (4)$$

where I_N is the $2^N \times 2^N$ identity matrix. It follows that the central quotient $P_N/Z(P_N)$ is an abelian group. The elements of $P_N/Z(P_N)$ are equivalence classes of the form

$$\{sA_1A_2\dots A_N \mid s \in \mu_4\} = \{\pm A_1A_2\dots A_N, \pm iA_1A_2\dots A_N\}, \quad (5)$$

and from each class we choose the N -qubit Pauli operator $A_1A_2\dots A_N$ as the representative.

Now we specialize to P_3 , but we note that most of our remarks can be easily reformulated for the general case. We are mainly interested in two aspects of P_3 . First, P_3 has the structure of a symplectic vector space over \mathbb{Z}_2 . Second, P_3 induces a symplectic polar space $W_5(2)$, which eventually leads us to what we call the *three-qubit Veldkamp space* whose special class of lines is the main concern of this paper.

Let us start with the symplectic vector space structure of P_3 . By (1), an arbitrary group element $A = sA_1A_2A_3 \in P_3$ can be written as

$$A = \left(i^{\sum_i a_i b_i}\right) sX^{a_1}Z^{b_1} \otimes X^{a_2}Z^{b_2} \otimes X^{a_3}Z^{b_3}, \quad (6)$$

where the exponents of the Pauli matrices take their value from $\{0, 1\} \subseteq \mathbb{N}$. (6) shows that A can be uniquely identified by the 7-tuple

$$(s, a_1, a_2, a_3, b_1, b_2, b_3). \quad (7)$$

By discarding the first element of (7), we get a vector

$$x = (a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{Z}_2^6, \quad (8)$$

that identifies the 3-qubit Pauli operator $A_1A_2A_3$, and consequently, the respective equivalence class of $P_3/Z(P_3)$. Because of $ZX = -XZ$ and $X^2 = Z^2 = I$, in the case of the matrix product AA' of $A, A' \in P_3$, (7) becomes

$$\left(ss'(-1)^{\sum_i a'_i b_i}, a_1 + a'_1, \dots, b_3 + b'_3\right), \quad (9)$$

where “+” means addition modulo 2, which is the addition of \mathbb{Z}_2 . Moreover, (9) implies that the group multiplication of $P_3/Z(P_3)$ induces the vector addition in \mathbb{Z}_2^6 .

(9) also implies that A and A' commute iff

$$\sum_{i=1}^3 (a_i b'_i + a'_i b_i) = 0. \quad (10)$$

The left hand side of (10) is a non-degenerate, skew-symmetric bilinear form on \mathbb{Z}_2^6 . In general, such bilinear forms are called *symplectic forms*, and vector spaces equipped with a symplectic form are called *symplectic vector spaces*. Thus \mathbb{Z}_2^6 with the symplectic form

$$\langle \cdot, \cdot \rangle : \mathbb{Z}_2^6 \times \mathbb{Z}_2^6 \rightarrow \mathbb{Z}_2, \quad (x, x') \mapsto \langle x, x' \rangle \equiv \sum_{i=1}^3 (a_i b'_i + b_i a'_i). \quad (11)$$

is a symplectic vector space over \mathbb{Z}_2 , which in the following will be denoted by V_3 . By a slight abuse of notation, we state the meaning of $\langle \cdot, \cdot \rangle$ as follows: given two equivalence classes of $P_3/Z(P_3)$ identified by $x, x' \in V_3$, an arbitrary element $A \in x$ commutes with an arbitrary element $A' \in x'$ iff $\langle x, x' \rangle = 0$. Often it is more convenient to write the vectors of V_3 as three-qubit Pauli operators instead of 6-tuples of 0's and 1's. For example, by (6), XYZ is equivalent to $(1, 1, 0, 0, 1, 1)$.

V_3 has special bases, also called *symplectic*, that are analogous to orthonormal bases of Euclidean spaces. Without loss of generality, we denote a symplectic basis of V_3 by

$$\{e_1, e_2, e_3, f_1, f_2, f_3\}, \quad (12)$$

and we assume that

$$\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0, \quad \langle e_i, f_j \rangle = \delta_{ij}, \quad i, j = 1, 2, 3. \quad (13)$$

The most important example of a symplectic basis is the *canonical basis* of V_3 :

$$\begin{aligned} e_1 &= XII = (1, 0, 0, 0, 0, 0), & f_1 &= ZII = (0, 0, 0, 1, 0, 0), \\ e_2 &= IXI = (0, 1, 0, 0, 0, 0), & f_2 &= IZI = (0, 0, 0, 0, 1, 0), \\ e_3 &= IIX = (0, 0, 1, 0, 0, 0), & f_3 &= IIZ = (0, 0, 0, 0, 0, 1). \end{aligned} \quad (14)$$

By letting $e_\mu = f_{\mu-3}$ for $\mu = 4, 5, 6$, the matrix of the symplectic form with respect to the canonical basis can be written as

$$J_{\mu\nu} \equiv \langle e_\mu, e_\nu \rangle = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}, \quad \mu, \nu = 1, 2, \dots, 6. \quad (15)$$

where 0 is the 3×3 zero matrix, and I_3 is the 3×3 identity matrix.

The invariance group of the symplectic form $\langle \cdot, \cdot \rangle$ is the group with matrix multiplication as group operation of the 6×6 matrices S that preserve $\langle \cdot, \cdot \rangle$, and consequently satisfy

$$S^T J S = J. \quad (16)$$

This group is called the *symplectic group of order 6 over \mathbb{Z}_2* , and it is denoted by

$$\mathrm{Sp}(6, \mathbb{Z}_2) \equiv \mathrm{Sp}(6, 2). \quad (17)$$

For each vector $p \in V_3$ there exists a *transvection* $T_p \in \mathrm{Sp}(6, 2)$ that acts on V_3 as follows:

$$T_p x = x + \langle x, p \rangle p, \quad x \in V_3. \quad (18)$$

It is known that the set of transvections generates $\mathrm{Sp}(6, 2)$ [27] and that there is a surjective homomorphism [26] from $W(E_7)$ i.e. the Weyl group of the exceptional group E_7 to $\mathrm{Sp}(6, 2)$ with kernel \mathbb{Z}_2 .

Next, we turn to the symplectic polar space realized by P_3 , and to its geometric hyperplanes, but first we need a few general definitions.

Definition. An *incidence structure* (or *point-line incidence geometry*) is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where \mathcal{P} and \mathcal{L} are disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is a relation, called the *incidence relation*. The elements of \mathcal{P} and \mathcal{L} are called *points* and *lines* respectively. We say that a point $p \in \mathcal{P}$ is *incident* with a line $l \in \mathcal{L}$ if $(p, l) \in \mathcal{I}$, and two points $p, p' \in \mathcal{P}$ are *collinear*, if both are incident with some line $l \in \mathcal{L}$.

In the following we limit ourselves to incidence structures whose lines are realized by subsets of the point set \mathcal{P} , and whose incidence relation is \in . In other words, $\mathcal{L} \subseteq 2^{\mathcal{P}}$, and $p \in \mathcal{P}$ is incident with $l \in \mathcal{L}$ iff $p \in l$. Such incidence structures $(\mathcal{P}, \mathcal{L}, \in)$ are called *simple*.

Next, we define special sets of points of an incidence structure, called geometric hyperplanes, that will be important later.

Definition. A subset $\mathcal{H} \subseteq \mathcal{P}$ is a *geometric hyperplane* of the incidence structure $(\mathcal{P}, \mathcal{L}, \in)$ if the following two conditions hold [22]:

(H1) $l \subseteq \mathcal{H}$ or $|\mathcal{H} \cap l| = 1$ for all $l \in \mathcal{L}$;

(H2) $\mathcal{H} \neq \mathcal{P}$.

Our aim is to examine the incidence structure that emerges from the N -qubit Pauli group P_N , and to study its geometric hyperplanes.

The projective space $PG(2N-1, 2)$ consists of the nonzero subspaces of the $2N$ dimensional vector space V_N over \mathbb{Z}_2 . The points of the projective space are one dimensional subspaces of the vector space, and more generally, k dimensional subspaces of the vector space are $k-1$ dimensional subspaces of the corresponding projective space. A subspace of $(V_N, \langle \cdot, \cdot \rangle)$ (and also the subspace in the corresponding projective space) is called *isotropic* if there is a vector in it which is orthogonal (with respect to the symplectic form) to the whole subspace, and *totally isotropic* if the subspace is orthogonal to itself. The space of totally isotropic subspaces of $(PG(2N-1, 2), \langle \cdot, \cdot \rangle)$ is called the *symplectic polar space* of rank $2N-1$ order two denoted by $\mathcal{W}(2N-1, 2)$. The maximal totally isotropic subspaces are called *Lagrangian subspaces*.

Let us illustrate these abstract concepts in terms of the physically relevant structures of three-qubit observables. The 63 nonzero vectors in V_3 are comprising the points of $PG(5, 2)$. Modulo an element of μ_4 they correspond to the nontrivial observables (i.e. the ones excluding the identity III). Let W be some subspace of $PG(5, 2)$ corresponding to a subset of nontrivial observables. Then W^\perp the orthogonal complement with respect to the symplectic form corresponds to the set of the observables *commuting* with the ones in W . A line $L \in PG(5, 2)$ is an object of the form $L = \lambda v + \mu u$ where $u, v \in PG(5, 2)$ and $\lambda, \mu \in \mathbb{Z}_2$ containing the three points: $u, v, u+v$. Hence a line corresponds to a triple of operators such that any two of them multiplto the third (up to an element of μ_4). An example of a line represented by three-qubit operators is $\{IIX, IYI, IIZ\}$. Now, the representative operators of a totally isotropic line, e.g. $\{IIX, IXI, IXX\}$, are also *pairwise commuting*. The number of totally isotropic lines is $315 = (64-1)(32-2)/6$.

A totally isotropic plane in $PG(5, 2)$ is of the form $P = \lambda v + \mu u + \nu w$ where u, v, w are linearly independent and pairwise commuting. Clearly, P contains seven points and seven totally isotropic lines. Since each line features three points and each point is incident with three lines, the incidence structure of the Fano plane, or $PG(2, 2)$ emerges[33]. As an example of a seven-tuple comprising the points of a Fano plane one can take the pairwise commuting set $\{IIX, IXI, IXX, XII, XIX, XXI, XXX\}$. The totally isotropic planes in $PG(5, 2)$ are maximal hence they are Lagrangian subspaces. It was shown[28] that the number of Lagrangian subspaces (which are in turn Fano planes) is 135. The geometry of the set of Lagrangian subspaces for three-qubits has been explored in[11]. In [11] in terms of the generators of a seven dimensional Clifford algebra an explicit list of all 135 Lagrangian subspaces was also presented. In summary: we see that $\mathcal{W}(5, 2)$ is comprising the 63 points the 315 totally isotropic lines and 135 totally isotropic planes of $PG(5, 2)$.

For an element $x \in V_3$ let us define the quadratic form

$$Q_0 \equiv \sum_{i=1}^3 a_i b_i. \quad (19)$$

It is easy to check that for vectors representing symmetric operators $Q_0(x) = 0$ (the ones containing an *even* number of Y s) and for antisymmetric ones $Q_0(x) = 1$ (the ones containing an *odd* number of Y s). Moreover we have the relation

$$\langle x, y \rangle = Q_0(x+y) + Q_0(x) + Q_0(y). \quad (20)$$

The (19) quadratic form will be regarded as a one labelled by the trivial element of V_3 with representative III . There are however, 63 other quadratic forms Q_p compatible the symplectic form $\langle \cdot, \cdot \rangle$ labelled by the nontrivial elements p of V_3 also satisfying

$$\langle x, y \rangle = Q_p(x+y) + Q_p(x) + Q_p(y). \quad (21)$$

They are defined as

$$Q_p(x) = Q_0(x) + \langle x, p \rangle^2 \quad (22)$$

since we are over the two element field the square can be omitted.

For more information on these quadratic forms we orient the reader to [28, 25]. Here we merely elaborate on the important fact that there are two classes of such quadratic forms. They are the ones that are labelled by symmetric observables ($Q_0(p) = 0$), and antisymmetric ones ($Q_0(p) = 1$). The locus of points in $PG(5, 2)$ satisfying $Q_p(x) = 0$ for $Q_0(p) = 0$ is called a *hyperbolic* quadric and the locus $Q_p(x) = 0$ for which $Q_0(p) = 1$ is called an *elliptic* one. The former one will be denoted by $Q^+(5, 2)$ and the latter by $Q^-(5, 2)$. Looking at (22) one can see that in terms of three-qubit observables (modulo elements of μ_4) one can characterize the quadrics $Q(5, 2)$ as follows. The three-qubit observables $x \in Q(5, 2)$ characterized by $Q_p(x) = 0$ are the ones that are either symmetric and commuting with p or antisymmetric and anticommuting with p . It can be shown [25, 28] that we have 36 quadrics of type $Q^+(5, 2)$ and 28 ones of type $Q^-(5, 2)$, with the former containing 35 and the latter containing 27 points of $PG(5, 2)$. A hyperbolic quadric of $Q^+(5, 2)$ type in $PG(5, 2)$ is called the *Klein-quadric*. On the other hand an elliptic quadric of $Q^-(5, 2)$ type can be shown to display the structure of a generalized quadrangle [29] $GQ(2, 4)$ an object we already mentioned in the introduction. A pictorial representation of this incidence structure having 27 points and 45 lines labelled by three-qubit observables can be found in [15].

Now we are ready to present the point-line incidence structure of N -qubit observables.

Definition. Let $N \in \mathbb{N} + 1$ be a positive integer, and V_N be the symplectic \mathbb{Z}_2 -linear space. The incidence structure \mathcal{G}_N of the N -qubit Pauli group is $(\mathcal{P}, \mathcal{L}, \in)$ where $\mathcal{P} = V_N \setminus \{0\}$,

$$\mathcal{L} = \{\{a, b, a + b\} | a, b \in \mathcal{P}, a \neq b, \langle a, b \rangle = 0\} \quad (23)$$

and \in is the set theoretic membership relation.

Clearly the points and lines of \mathcal{G}_N are the ones of the symplectic polar space $\mathcal{W}(2N-1, 2)$. Of course our main concern here is the $N = 3$ case. In this case \mathcal{G}_3 has 63 points and 315 lines.

Our next task is to recall the properties of the geometric hyperplanes of \mathcal{G}_N . The following lemma was proved in [25].

Lemma 1. Let $N \in \mathbb{N} + 1$ be a positive integer, $\mathcal{G}_N = (\mathcal{P}, \mathcal{L}, \in)$ and $p \in V_N$ be any vector. Then the following sets satisfy (H1):

$$C_p = \{x \in \mathcal{P} | \langle p, x \rangle = 0\}, \quad (24)$$

$$H_p = \{x \in \mathcal{P} | Q_p(x) = 0\}. \quad (25)$$

This lemma shows that apart from C_0 all of the sets above are geometric hyperplanes of the geometry \mathcal{G}_N . The set C_p is called the *perp set* of $p \in V_N$. Modulo an element of μ_4 C_p represents the set of observables commuting with a fixed one p . Back to the implications of our lemma one can show that in fact more is true, *all geometric hyperplanes* arise in this form [25]:

Theorem 1. Let $N \in \mathbb{N} + 1$, $\mathcal{G}_n = (\mathcal{P}, \mathcal{L}, \in)$, and $\mathcal{H} \in \mathcal{P}$ a subset satisfying (H1). Then either $\mathcal{H} = C_p$ or $\mathcal{H} = H_p$ for some $p \in V_N$.

One can prove that for $N \geq 2$ no geometric hyperplane is contained in another one, more precisely [25]

Theorem 2. Let $N \in \mathbb{N} + 2$, $\mathcal{G}_n = (\mathcal{P}, \mathcal{L}, \in)$ and suppose that $A, B \subset \mathcal{P}$ are two geometric hyperplanes. Then $A \subseteq B$ implies $A = B$.

Another property of two different geometric hyperplanes is that the complement of their symmetric difference gives rise to a third geometric hyperplane i.e.

Lemma 2. *For $A \neq B$ geometric hyperplanes in $\mathcal{G}_N = (\mathcal{P}, \mathcal{L}, \in)$ with $N \geq 1$ the set*

$$A \boxplus B := \overline{A \Delta B} = (A \cap B) \cup (\overline{A} \cap \overline{B}) \quad (26)$$

is also a geometric hyperplane.

One can also check that by using the notation $C \equiv A \boxplus B$

$$A \cap C = A \cap B, \quad B \cap C = A \cap B, \quad A \boxplus C = B. \quad (27)$$

A corollary of this is that any two of the triple $(A, B, A \boxplus B)$ of hyperplanes determines the third.

Sometimes it is also possible to associate to a particular incidence geometry another one called its Veldkamp space whose points are geometric hyperplanes of the original geometry[22]:

Definition. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a point-line geometry. We say that Γ has *Veldkamp points* and *Veldkamp lines* if it satisfies the conditions

(V1) For any hyperplane A it is not properly contained in any other hyperplane B .

(V2) For any three distinct hyperplanes A, B and C , $A \cap B \subseteq C$ implies $A \cap B = A \cap C$.

If Γ has Veldkamp points and Veldkamp lines, then we can form the Veldkamp space $V(\Gamma) = (\mathcal{P}_V, \mathcal{L}_V, \supseteq)$ of Γ , where \mathcal{P}_V is the set of geometric hyperplanes of Γ , and \mathcal{L}_V is the set of intersections of pairs of distinct hyperplanes.

Clearly, by Theorem 2, \mathcal{G}_N contains Veldkamp points for $N \geq 2$ hence in this case V1 is satisfied. In order to see that V2 holds as well we note[25]

Lemma 3. *Let $N \in \mathbb{N} + 1$, $a, b \in V_N$ and $\mathcal{G}_N = (\mathcal{P}, \mathcal{L}, \in)$. Then the following formulas hold:*

$$\begin{aligned} C_a \boxplus C_b &= C_{a+b}, \\ H_a \boxplus H_b &= C_{a+b}, \\ C_a \boxplus H_b &= H_{a+b}. \end{aligned} \quad (28)$$

From this it follows that for any three geometric hyperplanes A, B, C we have $A \cap B = A \cap C = B \cap C$. One can however show more[25], namely that there is no other possibility i.e. $A \cap B \subseteq C$ implies $C \in \{A, B, A \boxplus B\}$.

Theorem 3. *Let $N \in \mathbb{N} + 3$, and suppose that A, B, C are distinct geometric hyperplanes of $\mathcal{G}_n = (\mathcal{P}, \mathcal{L}, \in)$ such that $A \cap B \subseteq C$. Then $A \cap B = A \cap C$.*

Notice that the statement is not true for $N = 2$.

From these results it follows that there are two types of Veldkamp lines incident with three C -hyperplanes and three types of lines incident with one C -hyperplane and two H -hyperplanes. The former two types arise as a and b from C_a and C_b satisfy either $\langle a, b \rangle = 0$ or $\langle a, b \rangle = 1$. For the three types featuring two H -type hyperplanes are

$$\{\{H_a, H_b\} \mid a, b \in V_n, a \neq b, Q_0(a) = Q_0(b) = 0\}, \quad (29)$$

$$\{\{H_a, H_b\} \mid a, b \in V_n, a \neq b, Q_0(a) = Q_0(b) = 1\}, \quad (30)$$

$$\{\{H_a, H_b\} \mid a, b \in V_n, Q_0(a) \neq Q_0(b)\}. \quad (31)$$

The third type will be of great importance for us. For $N = 3$ this case gives rise to a Veldkamp line featuring a hyperbolic quadric, an elliptic quadric and a perp set. A canonical representation for this line can be chosen as $\{H_{III}, H_{YYY}, C_{YYY}\}$. An important result of [25] is the statement that $\text{Sp}(2N, 2)$ acts transitively on each of the previous three sets of doublets of geometric hyperplanes of H -type. For $N = 3$ we have 36 possibilities for choosing the hyperbolic and 28 ones for choosing the elliptic one. Hence in the class with canonical representative $\{H_{III}, H_{YYY}, C_{YYY}\}$ we have altogether $36 \cdot 28 = 1008$ Veldkamp lines. Then, by the transitivity of $\text{Sp}(6, 2)$, we can reach any of the 1008 Veldkamp lines from the canonical one via a set of suitable symplectic transvections of the form (18). For the construction of the explicit form of such transvections see [25]. A detailed description of these Veldkamp lines which comprise the Veldkamp space for the incidence structure \mathcal{G}_N can be found in Table 2. of [25]. Here, apart from the list of Veldkamp lines, the Reader will find the cardinalities of the core sets (intersection sizes of the three hyperplanes featuring the Veldkamp lines) and the number of copies of a particular type of Veldkamp lines. Since in the following we only need the $N = 3$ case and the aforementioned class of cardinality 1008, we refrain from presenting more details on these interesting issues.

3 Mermin pentagrams and their double-sixes

Mermin's pentagram is a set of ten observables, more specifically, nontrivial three-qubit Pauli-operators, that is used to prove the Bell–KS theorem in eight dimensions, i.e. to rule out non-contextual hidden-variables theories[6]. These observables are further arranged into five sets, or, as we like to call them, *lines* (see ‘incidence structure’ introduced in Section 2).

(P1) Each line contains, or is incident with four pairwise commuting operators that multiply to $\pm III$, hence one can speak of *positive* and *negative lines*.

Mermin's pentagram, published in [6], is reproduced in Figure 1. The configuration has four positive lines and a negative one; in the figure the latter is highlighted in boldface.

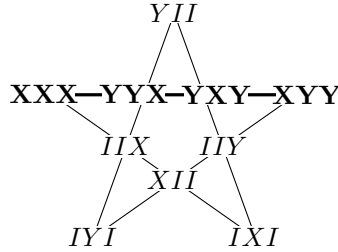


Figure 1: Mermin's pentagram

The other properties relevant to the proof of the Bell–KS theorem are as follows[9]:

(P2) the number of lines each observable is incident with is even (two);

(P3) the number of negative lines is odd.

By definition, a non-contextual hidden-variables theory is supposed to assign a value $v(A)$ to each possible observable A , such that $v(A)$ is an eigenvalue of A and

$$v(AB \dots) = v(A)v(B) \dots \quad (32)$$

holds for any pairwise commuting observables A, B, \dots . Property (P1) means that for any line $\{A_i, A_j, A_k, A_l\}$ of the pentagram we have

$$A_i A_j A_k A_l = \pm III. \quad (33)$$

Since the only eigenvalue of the observable $\pm III$ is ± 1 , $v(\pm III) = \pm 1$. Therefore, the values $v(A_i)$, $v(A_j)$, $v(A_k)$, $v(A_l)$ assigned to the observables of any line $\{A_i, A_j, A_k, A_l\}$ of the pentagram must satisfy

$$\pm 1 = v(\pm III) = v(A_i A_j A_k A_l) = v(A_i) v(A_j) v(A_k) v(A_l). \quad (34)$$

In the case of a positive line and a negative line, the LHS is $+1$ and -1 , respectively. Now, consider the product of (34) for all five lines of the pentagram. The LHS multiplies to -1 due to (P3). The product of the RHS is of the form

$$v(A_1)^{a_1} v(A_2)^{a_2} \dots v(A_{10})^{a_{10}} = +1, \quad (35)$$

since $v(A_i) = \pm 1$, and since the exponents a_i are even due to (P2). Thus we reach a contradiction. For further details we refer the reader to [4], Chapter 7.

We call a *Mermin pentagram*, or just simply a *pentagram* any set of ten three-qubit Pauli operators that can be arranged as the observables shown in Figure 1 and have properties (P1)-(P3) described above. Property (P2) is inherent to the pentagram-like arrangement. We know that (P3) follows from the rest[31], and that there are 7884, 4104 and 108 Mermin pentagrams with one, three and five negative lines respectively[10].

In Section 2 we saw that the three-qubit Pauli operators can be identified with the elements of the symplectic vector space V_3 over \mathbb{Z}_2 such that the multiplication of Pauli operators modulo $Z(P_3)$ (see (4)) corresponds to the vector addition in V_3 (see (9)), and the commutation of two operators corresponds to orthogonality with respect to the symplectic form $\langle \cdot, \cdot \rangle$ of (11). In this Section and the next we will treat Pauli operators as vectors of V_3 . Although by doing this we cut ourselves off from discussing property P3 however, for Mermin pentagrams this property is automatically satisfied[31] for any consistent three-qubit labelling. Actually more is true: for a pentagram regarded as a subgeometry of a multi-qubit symplectic polar space any consistent labelling guarantees that there are an odd number of negative lines (contexts)[31].

An important observation¹ is that the six Pauli operators *not* incident with an arbitrary line l of a Mermin pentagram form a symplectic basis of V_3 (see (12)). Moreover, since l shares exactly one operator with each line $l' \neq l$, due to (P1) the shared operator is determined by the other three incident with l' . Thus pentagrams can be constructed from symplectic bases of V_3 . However, this construction is not without ambiguity: each symplectic basis determines *two* pentagrams. For instance, the pentagrams constructed from the canonical basis (14) are shown in Figure 2. Notice that the arrangements of the basis vectors differ only by an exchange of XII and ZII . We say that these two pentagrams are each others *conjugate*.

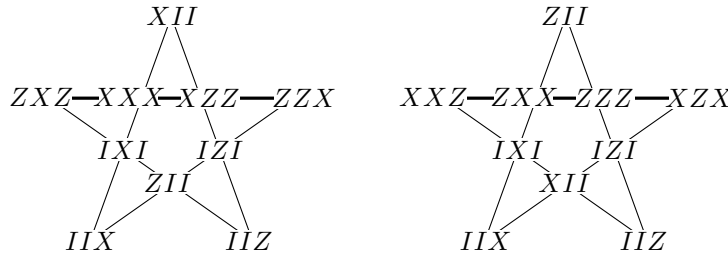


Figure 2: Mermin pentagrams constructed from the canonical symplectic basis of V_3

In general, every Mermin pentagram P has five conjugates, and each conjugate is associated with the line of P incident with the four observables not shared with the respective

¹This follows from properties (P1) and (P2) of the pentagrams. The argument can be found in [32].

conjugate. Thus the space of Mermin pentagrams can be regarded as a regular graph of degree five, where the pentagrams play the rôle of vertices and two pentagrams are adjacent iff they are conjugates. A computer program prepared to explore this graph revealed 1008 connected components, all isomorphic to the intersection graph of Schläfli's double six configuration. The latter graph is shown in Figure 3 (cf. [33], Section 4.5), and, to simplify the discussion, we will refer to it as *double six*.

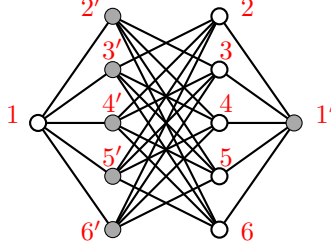


Figure 3: Intersection graph of Schläfli's double six, or ‘double six’ in our terminology

The double six is a bipartite graph with 12 vertices and 30 edges, and each vertex v has an *antipodal* vertex v' such that v and v' are not adjacent and belong to different partitions. The double six of pentagrams that contains the two pentagrams in Figure 2, obtained from the canonical basis, can be seen in Figure 4. Instead of conjugates, Figure 4 emphasizes the two partitions, the common sets of four observables forming a line (solid curves) and the antipodal pentagrams (the ones connected by dashed lines). In Figure 4 we can see a total of 20 observables, and these are already contained by any pair of antipodal pentagrams.

The product modulo μ_4 , or, in terms of the symplectic vector space V_3 , the sum of the ten observables² of some pentagram does not depend on the choice of pentagram, and it is specific to the double six. Let us denote this observable by w . In the case of the example in Figure 4, $w = YYY$. w anticommutes with all 20 observables of the double six, and the antipodal of any pentagram P is the image of P under the transvection T_w . The former observation implies that the restriction of T_w to the 20 observables of the double six is the addition of w .

Let $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ be a symplectic basis the elements of which are not incident with an arbitrary line of an arbitrary pentagram of the double six, and let

$$p = \sum_{i=1}^3 [Q_0(e_i)f_i + Q_0(f_i)e_i]. \quad (36)$$

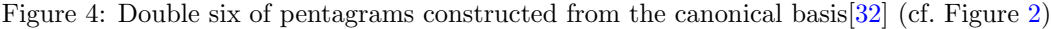
We assert without proof³ that p is also specific to the double six, i.e. it does not depend on the choice of pentagram and line, it is symmetric and $q = p + w$ is antisymmetric. It is easy to check that in our case the double six is characterized by $p = III$ and $q = YYY$. Moreover, the pair p, q unambiguously identifies the double six. Since the number of symmetric and antisymmetric three-qubit Pauli operators is 36 and 28 respectively, we get $1008 = 36 \cdot 28$ double sixes of pentagrams. These results will be clear when we relate them to properties of a special Veldkamp line in Section 5.

4 Double-sixes connected to spreads of the “doily”

The Lagrangian subspaces of V_3 were introduced in Section 2, and we saw that these subspaces correspond to maximal sets of pairwise commuting three-qubit Pauli operators. Sup-

²Actually, it suffices to consider only six observables not incident with some line.

³Some of the arguments can be found in [32].



pose we have a Mermin pentagram P and let l_i , $i = 1, \dots, 5$ denote its lines, i.e. the sets of four pairwise orthogonal vectors, realized of course by four pairwise commuting observables. Any three vectors incident with a line l_i are linearly independent, thus l_i spans a Lagrangian subspace U_i . If, say, $l_i = \{a, b, c, d\}$, then d , chosen arbitrarily, can be written as

$$d = a + b + c, \quad (37)$$

and the set of nontrivial vectors in U_i is of the form

$$\{a, b, c, a + b, b + c, a + c, a + b + c\}. \quad (38)$$

The set of three nontrivial vectors

$$L_i = \{a + b, b + c, a + c\} \subseteq U_i \setminus l_i \quad (39)$$

is a totally isotropic line of $\mathcal{W}(5, 2)$, and each of its elements, being the sum of two vectors *not* orthogonal to w (see Section 3), is orthogonal to w .

The sets L_i are mutually disjoint, thus $\{L_1, L_2, L_3, L_4, L_5\}$ is a partition of $\mathcal{P} \equiv \bigcup L_i$. By looking at a different pentagram P' from the same double six P belongs to, one gets the same set \mathcal{P} of 15 observables. Moreover, if P' is the antipodal of P , even the partition of \mathcal{P} will be the same, i.e. one gets

$$\{L_1, L_2, L_3, L_4, L_5\} = \{L'_1, L'_2, L'_3, L'_4, L'_5\}. \quad (40)$$

The incidence structure $(\mathcal{P}, \mathcal{L}, \in)$, where \mathcal{L} is the set of every possible set L_i for the respective double six, is a subgeometry of $\mathcal{W}(5, 2)$ well known to finite geometers; it is called the *doily* (Figure 5, left; for more details about the doily see [33], Section 4.1).

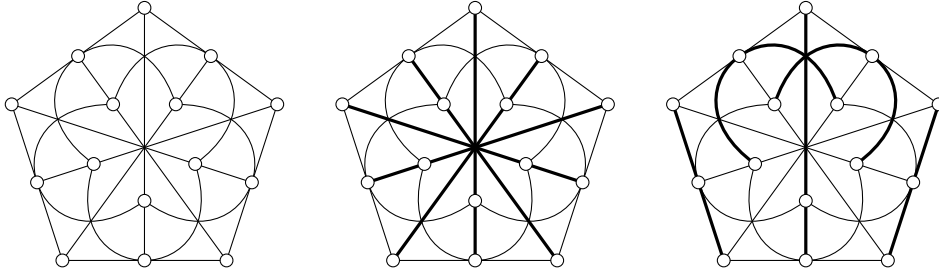


Figure 5: The doily and two of its spreads

By definition, a set of lines that is a partition of \mathcal{P} is a *spread* of the doily, and there are a total of six spreads. Two of these are shown in Figure 5, middle and right, where the lines contained are highlighted in thick curves, and the remaining four can be depicted as in the rightmost diagram of Figure 5 up to a rotation by $n \cdot 72$ degrees, $n = 1, 2, 3, 4$. Thus, each pair P, P' of antipodal pentagrams of the double six is related to a spread of the doily. For the case of the double six in Figure 4 this relation is illustrated in Figure 6. Figure 6 shows only one of the two partitions, and the large circles filled with gray represent points of the doily. The thickness of the curves does not represent anything, it only helps to distinguish the relevant objects.

5 Double-sixes of pentagrams arising from Veldkamp lines

We have seen that 12 Mermin pentagrams featuring a set of 20 three-qubit observables can be organized into a $6 + 6$ configuration dubbed "double-six", which are in turn connected to

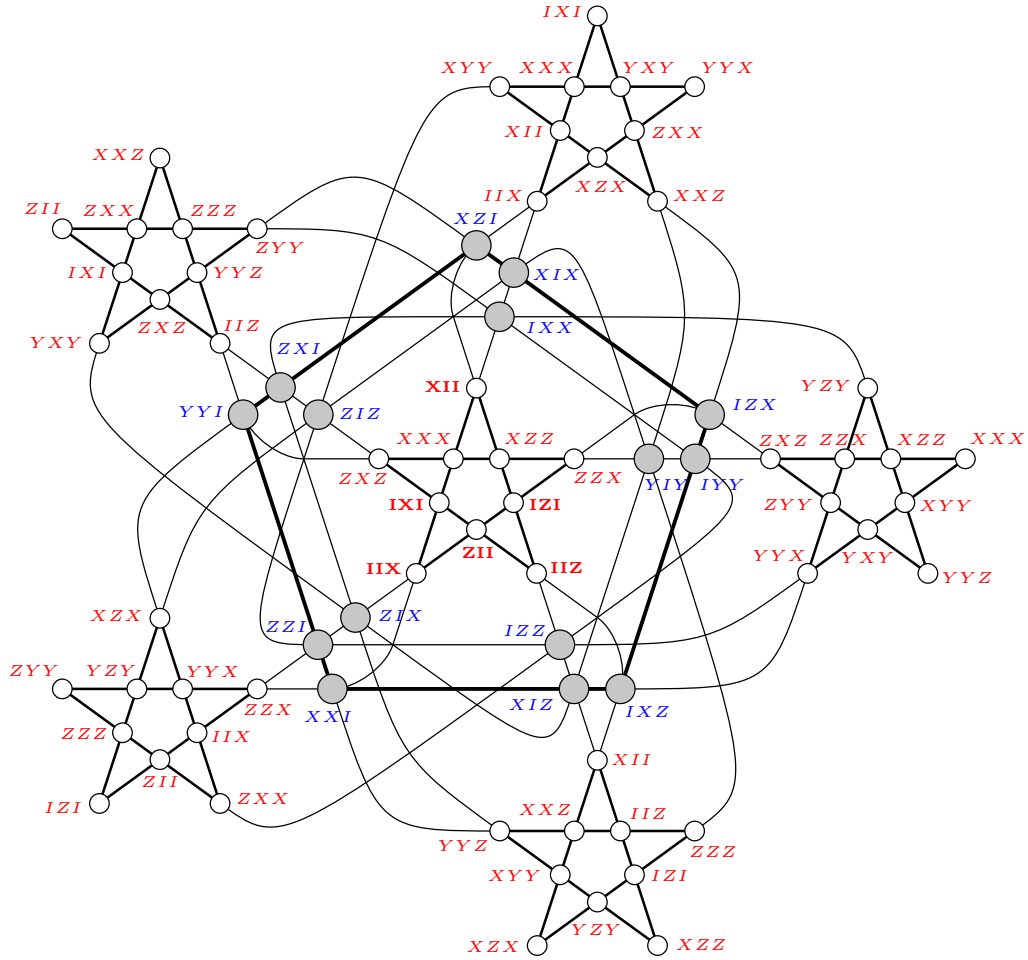


Figure 6: The doily and one of the two partitions of the double six

spreads of a single "doily" a generalized quadrangle $GQ(2, 2)$ featuring a set of 15 further observables. The connection between these incidence structures was effected by the structure of a set of Lagrangian subspaces, representing seven-tuples of mutually commuting sets of observables. Indeed, this 15+20 split of observables yielded a cut of the relevant Lagrangian subspaces into a 3+4 structure, where the triples and quadruplets of commuting observables were residing within the doily and the double-six of pentagrams respectively.

Another crucial observation is that all of these $35 = 15 + 20$ observables are *symmetric* ones, meaning that the corresponding points in $\mathcal{W}(5, 2)$ are lying on the hyperbolic Klein quadric. In other words these 35 points are lying on a geometric hyperplane of type H_{III} .

We have also seen that in this picture a further observable namely YYY acquired a special status. The transvection T_{YYY} was exchanging the pentagrams taken from one of the sets of six pentagrams with the ones of the other set of six ones. Moreover, YYY was commuting with all of the 15 observables comprising the doily. Hence the 15 observables of H_{III} can also be regarded as the ones taken from the 31 element perp set of another geometric hyperplane namely C_{YYY} .

One further piece of evidence comes from [14]. In this work a labelling of the 27 points of the generalized quadrangle $GQ(2, 4)$ (alias an elliptic quadric in $\mathcal{W}(5, 2)$) in terms of three-qubit observables have been given. According to Figure 3 of [14], the 27 points split as $15 + 6 + 6$, where the 15 element set again gives rise to the incidence structure of a doily. The $6 + 6$ configuration is isomorphic to the double six shown in Figure 3[14], and the points of this double-six are interchanged via a transvection T_w where w corresponds to the point defining the elliptic quadric, i.e. $Q_w(x) = 0$. Taking this point to be $w = YYY$ we obtain yet another geometric hyperplane H_{YYY} and one can easily check that for this choice the 15 points will be precisely the 15 ones we have already come across with in connection with the hyperplanes C_{YYY} and H_{III} .

From these considerations it is clear that what we have described here is just the Veldkamp line $(H_{III}, H_{YYY}, C_{YYY})$ whose core set is the doily labelled with a particular set of symmetric three-qubit observables.

In order to make these considerations very explicit, in the following we give the observables representing the points of the geometric hyperplanes comprising our particular Veldkamp line. First, we list the observables of each hyperplane that *do not belong* to the common core set. The 20 points taken from the Klein-quadric H_{III} are the symmetric observables

$$\begin{array}{cccccccccc} XXX & XXZ & XZX & ZXX & IIX & IXI & XII & IIZ & IZI & ZII \\ ZZZ & ZZX & ZXZ & XZZ & YYZ & YZY & ZYY & YYX & YXY & XYY \end{array} \quad (41)$$

which build up the double-six of pentagrams, where the $10 + 10$ split refers to the role of the transvection T_{YYY} exchanging the sets. The 12 points taken from the elliptic quadric H_{YYY} are comprising the antisymmetric observables anticommuting with YYY (see (22)). These are

$$\begin{array}{cccccc} YXI & IYZ & IYX & XIY & ZIY & YZI \\ IZY & YIX & YIZ & ZYI & XZI & IXY \end{array} \quad (42)$$

where the $6 + 6$ split refers to Schläfli's double-six configuration with the transvection T_{YYY} exchanging the sets. The 16 points taken from the perp set C_{YYY} are comprising the antisymmetric observables commuting with YYY . These are

$$\begin{array}{cccccc} YYY & IYY & IYI & YII & ZZY & ZYZ & YZZ & XXY & XYX & YXX \\ XYZ & ZXY & YZX & ZYX & XZY & YXZ \end{array} \quad (43)$$

And finally, the 15 points of the common core set are

$$\begin{array}{cccccc} XXI & XIX & IXX & YYI & YIY & IYY & ZZI & ZIZ & IZZ & IXZ \\ ZIX & XZI & ZXI & IZX & XIZ \end{array} \quad (44)$$

These observables belong to all of the hyperplanes $H_{III}, H_{YYY}, C_{YYY}$ and can be used to label the incidence structure of a doily. Indeed, these observables are symmetric (corresponding to points in H_{III}), they are symmetric and commuting with YYY (corresponding to points of H_{YYY}), and finally they are commuting with YYY (corresponding to points of C_{YYY}). Altogether we have $63 = 20 + 12 + 16 + 15$ nontrivial observables (modulo elements of μ_4) yielding the correct number of points of $\mathcal{W}(5, 2)$. This structure of our Veldkamp line is highlighted in Figure 7.

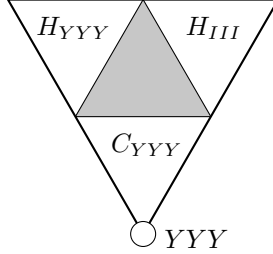


Figure 7: The Veldkamp line $H_{III} - H_{YYY} - C_{YYY}$

We already know[25] that the number of Veldkamp lines featuring a hyperbolic and an elliptic quadric is $36 \cdot 28 = 1008$. Each Veldkamp line is associated with a double-six configuration of pentagrams. The structure of a double-six configuration is entirely encoded into the structure of spreads of the core set.

Clearly, no two double-six configurations associated with *different* Veldkamp lines contain the same pentagram. Indeed, if we assume the existence of such two pentagrams, they should give rise to the same spreads, hence the same core sets i.e. doilies. Hence according to property V2 this contradicts our assumption of different Veldkamp lines. In short, the core set is unambiguously characterizes the Veldkamp line hence the total number of pentagrams is $12 \cdot 1008 = 12096$. This argument provides a clear geometric understanding of the computer aided result of [10]. However, apart from our argument producing the right number of pentagrams in a simple manner, it also provides considerable amount of new geometric data on the structure of pentagrams. More importantly for the first time our result establishes a clear physical meaning of the abstract mathematical notion of a Veldkamp line featuring elliptic and hyperbolic quadrics: namely it encodes information on the geometry of all possible Mermin pentagrams that can be built from three-qubit observables.

Let us also show that our argument also produces an explicit algorithm for generating all possible double-sixes from a particular one. Indeed, let us call the Veldkamp line $(H_{III}, H_{YYY}, C_{YYY})$ the *canonical Veldkamp line*: \mathcal{L}_V^* . According to [25] we have

Lemma 4. *Let $N \in \mathbb{N} + 3$, $\mathcal{G}_N = (\mathcal{P}, \mathcal{L}, \epsilon)$ and $a, b, f \in V_N$ three distinct vectors such that $Q_0(a) = Q_0(b)$. Then there exists an element in $\text{Sp}(2N, 2)$ fixing H_f and swapping H_a with H_b .*

Using this result one can conclude that $\text{Sp}(2N, 2)$ acts transitively on the set of Veldkamp lines featuring two H -type hyperplanes. In our particular case this means that for any two Veldkamp lines from the 1008 ones one can find an element of $\text{Sp}(6, 2)$ transforming them to each other. The proof of this Lemma constructs this element of $\text{Sp}(6, 2)$ by giving the explicit form of the involutions swapping the hyperplanes of the same type.

Since we will use it later let us describe this construction (for a proof see [25]). If $Q_f(a + b) = 1$, the transvection T_{a+b} is an involution, swaps H_a and H_b , and fixes H_f . If $Q_f(a + b) = 0$, pick an element $p \in C_{a+b} \cap H_a \setminus H_f$, and form $q = p + a + b$. Then T_p and T_q commute, hence $T_p T_q$ is an involution, moreover, $T_p T_q H_a = H_b$, and H_f is again fixed.

As an example needed later we take the following two Veldkamp lines: $(H_{III}, H_{YYY}, C_{YYY})$ and $(H_{YI}, H_{YYY}, C_{IYY})$. The first one corresponds to our canonical line \mathcal{L}_V^* . We would like to transform this line to the new one. Now $f = YYY$, $a = III$ and $b = YI$. Since $Q_f(a+b) = 0$ we have to find a $p \in C_{YI} \cap H_{III} \setminus H_{YYY}$. This means that we have to find a symmetric observable commuting with YI and anticommuting with YYY . As a particular choice let us take $p = XXX$, hence $q = ZZX$. Then (among the many possible involutions) an involution that gives us the job is $T_{ZZX}T_{XXX}$.

Let us group the 20 observables of (41) to a $1 + 9 + 9 + 1$ split. The meaning of this split will be clarified later. Then the 20 new observables belonging to the new hyperbolic quadric H_{YI} are

$$T_{ZZX}T_{XXX} \begin{pmatrix} XXX & IZI & ZZX \\ ZII & YYX & XII \\ ZXZ & IXI & ZZX \end{pmatrix} = \begin{pmatrix} XXX & XYX & XZX \\ YXX & YYX & YZX \\ ZXX & ZYX & ZZX \end{pmatrix} \quad (45)$$

$$T_{ZZX}T_{XXX} \begin{pmatrix} ZZZ & YXY & XZZ \\ XYY & IIZ & ZYY \\ ZXZ & YZY & XZX \end{pmatrix} = \begin{pmatrix} XXZ & XYZ & XZZ \\ YXZ & YYZ & YZZ \\ ZXZ & ZYZ & ZZZ \end{pmatrix} \quad (46)$$

$$T_{ZZX}T_{XXX}IIX = IIX, \quad T_{ZZX}T_{XXX}YYZ = IIZ. \quad (47)$$

For the transformed observables the pattern should be clear: the two groups of $1 + 9 = 10$ observables are distinguished by the third qubit observable being either X or Z . Apart from the observables IIX and IIZ the remaining $9 + 9$ ones are organized into 3×3 matrices with the rows and columns are labelled by the first and second qubit observables taking the values: X, Y, Z . Since the involutions $T_{ZZX}T_{XXX}$ leave the symplectic form invariant, the commutation properties are left invariant as well. As a result this involution transforms the canonical double-six configuration to a new one featuring a new set of 12 pentagrams. Repeating these steps one can reach any double-six hence any of the 12096 pentagrams.

6 The representation theoretic meaning of double-sixes

According to the results of the previous section from the canonical double-six configuration one can generate any such configuration. Hence we are left with the clarification of the geometry of this canonical configuration.

The double six configuration is featuring 20 observables. According to (45)-(47) these observables can be organized into a $1 + 9 + 9 + 1$ structure reminiscent of the representation theoretic content of Freudenthal triple systems displaying a $1 + J + J + 1$ structure based on cubic Jordan algebras with $J = 6, 9, 15, 27$. Such systems have already been explored within the context of quantum entanglement[34, 35, 16, 36]. The particular values of J correspond to choosing the Jordan algebras as the algebras of Hermitian 3×3 matrices over the reals, complex numbers, quaternions and octonions. The numbers 6, 9, 15, 27 are the numbers of independent real parameters these matrices have. It can be shown that the $1 + J + J + 1$ dimensional module constructed in this way gives rise to irreducible representations of dimension 14, 20, 32, 56 of particular real forms of the groups of type C_3 , A_5 , D_6 , E_7 . In particular the $20 = \binom{6}{3}$ dimensional representation of a real form $SU(6)$ of the group A_5 is furnished by the space of real totally antisymmetric tensors (three forms) $P_{\mu\nu\rho}$ in a six dimensional space \mathbb{R}^6 ($\mu, \nu, \rho = 1, 2, \dots, 6$). The group action is

$$P_{\mu\nu\rho} \mapsto S_\mu^{\mu'} S_\nu^{\nu'} S_\rho^{\rho'} P_{\mu'\nu'\rho'}, \quad S_\mu^{\mu'} \in SU(6). \quad (48)$$

The 20 components of $P_{\mu\nu\rho}$ can be organized to two numbers and two 3×3 matrices as follows[34]. Let us adopt the new labelling $\{1, 2, 3, 4, 5, 6\} \equiv \{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$. Then the components of $P_{\mu\nu\rho}$ can be split to two 3×3 matrices and two numbers as follows

$$\begin{pmatrix} P_{123} & P_{131} & P_{112} \\ P_{223} & P_{231} & P_{212} \\ P_{323} & P_{331} & P_{312} \end{pmatrix}, \quad \begin{pmatrix} P_{1\bar{2}\bar{3}} & P_{1\bar{3}\bar{1}} & P_{1\bar{1}\bar{2}} \\ P_{2\bar{2}\bar{3}} & P_{2\bar{3}\bar{1}} & P_{2\bar{1}\bar{2}} \\ P_{3\bar{2}\bar{3}} & P_{3\bar{3}\bar{1}} & P_{3\bar{1}\bar{2}} \end{pmatrix}, \quad P_{1\bar{2}\bar{3}}, \quad P_{123}. \quad (49)$$

Now comparing the right hand sides of (45)-(47) with (49) one can conjecture that the geometry of the weight diagram of the 20 dimensional irrep of A_5 and the geometry of the double-sixes might be related. Indeed, the "overline" involution acting like $P_{123} \mapsto P_{1\bar{2}\bar{3}}$ in this picture seems to correspond to the involution $T_{I\bar{I}Y}$. Moreover, in this picture one can take the vector P_{123} as the one labelled by IIX which in turn should somehow correspond to the highest weight vector of the irrep 20 of $SU(6)$.

Via employing an explicit construction in the following we prove that the geometry of the double-six configuration can indeed be mapped to the geometry of the weight diagram of the 20 of $SU(6)$.

The group $SU(6)$ is one of the real forms of the group of type A_5 which has rank five. The five simple roots can be described[26] as the vectors living in a five dimensional hyperplane of \mathbb{R}^6 with normal vector $n = (111111)$ as follows

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_4 - e_5, \quad \alpha_5 = e_5 - e_6 \quad (50)$$

where the $e_\mu, \mu = 1, \dots, 6$ form the canonical orthonormal basis vectors of \mathbb{R}^6 . These vectors satisfy $(n, \alpha_\mu) = 0, \mu = 1, 2, 3, 4, 5$ ((\cdot, \cdot) is the ordinary scalar product in \mathbb{R}^6) and are linearly independent hence can be regarded as the basis vectors of the five dimensional hyperplane where the root vectors of A_5 reside. The length of the simple roots is 2, and they are either orthogonal or having an angle 120 degrees between them, hence the Cartan matrix[37] and its inverse for A_5 are

$$A_{ij} = (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad G_{ij} = \frac{1}{6} \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}. \quad (51)$$

An arbitrary weight vector Λ living in this five dimensional hyperplane can be expressed as

$$\Lambda = \sum_{i=1}^5 \lambda_i \alpha_i \quad (52)$$

where the λ_i are the components of the weight vector in the *dual* basis[37]. The *Dynkin labels* of Λ are given by the formula

$$a_i = \sum_{j=1}^5 \lambda_j A_{ji}. \quad (53)$$

Clearly the Dynkin labels of the α_i are given by the columns of the Cartan matrix hence

$$\alpha_1 = (2, -1, 0, 0, 0), \quad \dots, \quad \alpha_5 = (0, 0, 0, -1, 2). \quad (54)$$

This illustrates a theorem of crucial importance that the Dynkin labels of any root or weight are always *integers*. In the case of A, D, E type groups the scalar product between any weights can be expressed as[37]

$$(\Lambda, \Lambda') = a'_i G_{ij} a_j = \lambda'_i A_{ij} \lambda_j \quad (55)$$

where for simplicity summation was left implicit.

The Dynkin labels of the highest weight $\Lambda^{(0)}$ vector of the 20 dimensional irrep of A_5 is[37]

$$a_i^{(0)} = (0, 0, 1, 0, 0). \quad (56)$$

Since only the Dynkin label $a_3^{(0)}$ is different from zero from this highest weight one can subtract the simple root α_3 . As a result we get a new weight vector in the representation this time $\Lambda^{(1)}$ with Dynkin labels $a_i^{(1)} = (0, 1, -1, 1, 0)$. In the next step one can subtract α_2 and α_4 etc. As a result of this procedure one obtains the weight diagram for the 20 of A_5 see Figure 8.

In the following the 20 weight vectors of this representation will be labelled as follows: $\Lambda^{(0)}, \Lambda^{(r)}, \Lambda^{(\bar{r})} \Lambda^{(\bar{0})}$, where $r = 1, 2, \dots, 9$. This labelling is reminiscent of the $1 + 9 + 9 + 1$ structure we have already described. Let us denote the Dynkin labels accordingly as $a_i^{(0)}$, $a_i^{(r)}$, $a_i^{(\bar{r})}$, $a_i^{(\bar{0})}$ then we have a list

$$\begin{aligned} a_i^{(1)} &= (0, 1, -1, 1, 0), & a_i^{(2)} &= (1, -1, 0, 1, 0), & a_i^{(3)} &= (0, 1, 0, -1, 1), \\ a_i^{(4)} &= (-1, 0, 0, 1, 0), & a_i^{(5)} &= (1, -1, 1, -1, 1), & a_i^{(6)} &= (0, 1, 0, 0, -1), \\ a_i^{(7)} &= (-1, 0, 1, -1, 1), & a_i^{(8)} &= (-1, 0, 1, 0, -1), & a_i^{(9)} &= (1, -1, 1, 0, -1), \end{aligned} \quad (57)$$

where the remaining 10 weights are obtained from these ones via flipping the signs. For example $a_i^{(9)} = (-1, 1, -1, 0, 1)$ and $a_i^{(\bar{0})} = (0, 0, -1, 0, 0)$.

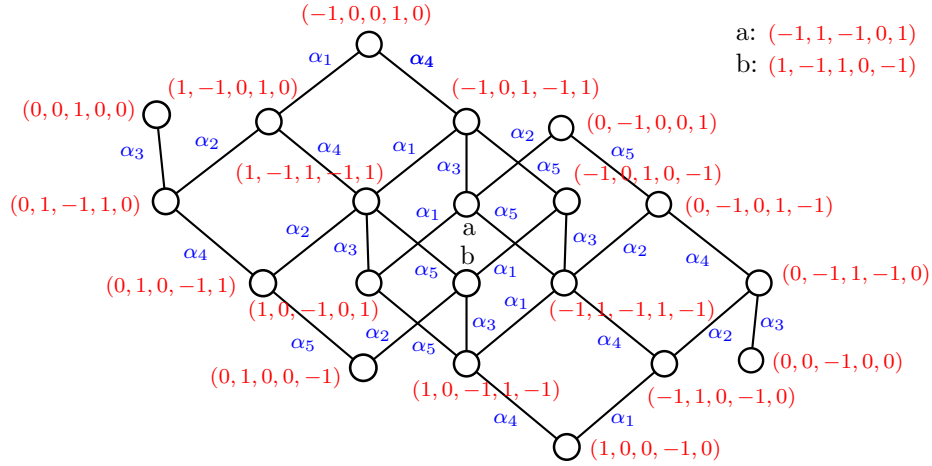


Figure 8: Weight diagram of the 20 dimensional irrep of A_5

As a next step inverting (53) we can calculate the components of these weights in the dual basis with the result: $\lambda_i^{(0)}, \lambda_i^{(r)}, \lambda_i^{(\bar{r})}, \lambda_i^{(\bar{0})}$. Using these components in (52) and expressing the simple roots in the (50) orthonormal basis one obtains the 20 weights as elements of \mathbb{R}^6

as follows

$$\begin{aligned}
\Lambda^{(0)} &= \frac{1}{2}(1, 1, 1, -1, -1, -1) \quad , \quad \Lambda^{(1)} = \frac{1}{2}(1, 1, -1, 1, -1, -1), \\
\Lambda^{(2)} &= \frac{1}{2}(1, -1, 1, 1, -1, -1), \quad \Lambda^{(3)} = \frac{1}{2}(1, 1, -1, -1, 1, -1), \\
\Lambda^{(4)} &= \frac{1}{2}(-1, 1, 1, 1, -1, -1), \quad \Lambda^{(5)} = \frac{1}{2}(1, -1, 1, -1, 1, -1), \\
\Lambda^{(6)} &= \frac{1}{2}(1, 1, -1, -1, -1, 1), \quad \Lambda^{(7)} = \frac{1}{2}(-1, 1, 1, -1, 1, -1), \\
\Lambda^{(8)} &= \frac{1}{2}(-1, 1, 1, -1, -1, 1), \quad \Lambda^{(9)} = \frac{1}{2}(1, -1, 1, -1, -1, 1).
\end{aligned} \tag{58}$$

and the remaining 10 weights $\Lambda^{(\bar{0})}, \Lambda^{(\bar{r})}$ are obtained by flipping the signs. Notice that all of these weight vectors are residing in the five dimensional hyperplane with normal vector n (the sum of the components is zero). The $20 = \binom{6}{3}$ vectors are arising from the possibilities of choosing from the six slots arbitrary three to have minus signs. All of these vectors are lying on the intersection of a five dimensional sphere of radius squared $\frac{3}{2}$ and our five dimensional hyperplane. Notice also that we have

$$(\Lambda^{(0)}, \Lambda^{(\bar{0})}) = (\Lambda^{(r)}, \Lambda^{(\bar{r})}) = -\frac{3}{2}, \quad r = 1, \dots, 9 \tag{59}$$

hence by virtue of

$$\cos(\Lambda^{(p)}, \Lambda^{(\bar{p})}) = \frac{(\Lambda^{(p)}, \Lambda^{(\bar{p})})}{\|\Lambda^{(p)}\| \|\Lambda^{(\bar{p})}\|} = -1, \quad p = 0, 1, \dots, 9 \tag{60}$$

these vectors are *antipodal*.

For the remaining pairs of vectors we have

$$|(\Lambda^{(p)}, \Lambda^{(q)})| = \frac{1}{2}, \quad p \neq \bar{q}, \quad p, q = 0, 1, \dots, 9, \bar{0}, \bar{1}, \dots, \bar{9}. \tag{61}$$

A much more compact labelling of the 20 weight vectors can be obtained by adopting the labelling of the six component vectors $\{1, 2, 3, 4, 5, 6\} \equiv \{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$ as in (49). By recording only those slots of the six component vectors which have *positive* entries one obtains a *trivector labelling* of the weight vectors. For example $\Lambda^{(1)}$ has positive entries in slots 124 i.e. $12\bar{1}$ after a permutation we refer to this weight as $\Lambda^{(\bar{1}12)}$. We employed this permutation in order to conform with the trivector labelling convention of (49). In this way we obtain the dictionary

$$\Lambda^{(0)} = \Lambda^{(123)}, \quad \begin{pmatrix} \Lambda^{(4)} & \Lambda^{(2)} & \Lambda^{(1)} \\ \Lambda^{(7)} & \Lambda^{(5)} & \Lambda^{(3)} \\ \Lambda^{(8)} & \Lambda^{(9)} & \Lambda^{(6)} \end{pmatrix} = \begin{pmatrix} \Lambda^{(\bar{1}23)} & \Lambda^{(\bar{1}31)} & \Lambda^{(\bar{1}12)} \\ \Lambda^{(\bar{2}23)} & \Lambda^{(\bar{2}31)} & \Lambda^{(\bar{2}12)} \\ \Lambda^{(\bar{3}23)} & \Lambda^{(\bar{3}31)} & \Lambda^{(\bar{3}12)} \end{pmatrix} \tag{62}$$

and the remaining 10 weights are obtained by applying the "overline" involution, e.g. $\Lambda^{(\bar{1})} = \Lambda^{(\bar{1}\bar{1}\bar{2})}$. Let us now denote by \mathcal{A}, \mathcal{B} three element subsets of the set $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$. Then the scalar products of the 20 weight vectors can be compactly summarized as

$$(\Lambda^{(\mathcal{A})}, \Lambda^{(\mathcal{B})}) = \begin{cases} -\frac{3}{2} & , \quad |\mathcal{A} \cap \mathcal{B}| = 0 \\ -\frac{1}{2} & , \quad |\mathcal{A} \cap \mathcal{B}| = 1 \\ +\frac{1}{2} & , \quad |\mathcal{A} \cap \mathcal{B}| = 2 \\ +\frac{3}{2} & , \quad |\mathcal{A} \cap \mathcal{B}| = 3. \end{cases} \tag{63}$$

Let us now try to map the 20 weight vectors of (62) to the 20 observables showing up in a double-six of pentagrams arising from the Veldkamp line $(H_{YYI}, H_{YYI}, C_{IYY})$. For this Veldkamp line we know that the right hand sides of (45)-(47) show similar structure to the one of weight vectors. Hence we define the correspondence

$$\Lambda^{(123)} \leftrightarrow IIX, \quad \begin{pmatrix} \Lambda^{(\bar{1}23)} & \Lambda^{(\bar{1}31)} & \Lambda^{(\bar{1}12)} \\ \Lambda^{(\bar{2}23)} & \Lambda^{(\bar{2}31)} & \Lambda^{(\bar{2}12)} \\ \Lambda^{(\bar{3}23)} & \Lambda^{(\bar{3}31)} & \Lambda^{(\bar{3}12)} \end{pmatrix} \leftrightarrow \begin{pmatrix} XXZ & XYZ & XZZ \\ YXZ & YYZ & YZZ \\ ZXZ & ZYZ & ZZZ \end{pmatrix} \quad (64)$$

and the correspondence for the remaining weights and observables is established by the action of the "overline" involution on the weight side corresponding to the transvection T_{IYY} on the observable side. Hence for example $\Lambda^{(1\bar{2}\bar{3})} \leftrightarrow XXX$ etc.

Let us consider now the (64) correspondence

$$\Lambda^{(\mathcal{A})} \leftrightarrow \mathcal{O}_{\mathcal{A}} \quad (65)$$

between weight vectors labelled by three element subsets of $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$ and the 20 observables of the three-qubit Pauli group (modulo elements of μ_4) featuring our double-six. It is easy to check that

$$\begin{aligned} |\mathcal{A} \cap \mathcal{B}| = \text{even} &\leftrightarrow \{\mathcal{O}_{\mathcal{A}}, \mathcal{O}_{\mathcal{B}}\} = 0, \\ |\mathcal{A} \cap \mathcal{B}| = \text{odd} &\leftrightarrow [\mathcal{O}_{\mathcal{A}}, \mathcal{O}_{\mathcal{B}}] = 0. \end{aligned} \quad (66)$$

i.e. if the intersection size of weight vector labels are even (odd) the corresponding observables are anticommuting (commuting). For example the weight labels $\{\bar{1}23\}$ and $\{\bar{1}31\}$ have intersection size 2 hence the corresponding observables XXZ and XYZ are anticommuting, or the weight labels $\{\bar{1}23\}$ and $\{\bar{2}31\}$ have intersection size 1 hence the observables XXZ and YYZ are commuting. As a last example note that the labels of the highest and lowest weight vectors i.e. $\{123\}$ and $\{\bar{1}\bar{2}\bar{3}\}$ are disjoint hence the corresponding observables IIX and IIZ are anticommuting. Clearly every observable is commuting with itself which corresponds to the fact that the intersection size equals 3.

Notice, that we are merely interested in whether two observables are commuting or anticommuting, which in the finite geometric context translates to the corresponding points being collinear or not. This information translates to the incidence structure between weight vectors having scalar product $-\frac{1}{2}, \frac{3}{2}$ (commuting), or $\frac{1}{2}, -\frac{3}{2}$ (not commuting). Since norm-squared for weight vectors equals $\frac{3}{2}$ the conditions for commuting of observables simplify to the single one that the corresponding weight vectors $\Lambda^{(\mathcal{A})}$ and $\Lambda^{(\mathcal{B})}$ are incident when the angle between them satisfies

$$\cos \theta_{\mathcal{AB}} = -\frac{1}{3} \quad (67)$$

which happens when $|\mathcal{A} \cap \mathcal{B}| = 1$.

Having clarified the meaning of commuting in the weight diagram picture, we should find a condition which corresponds to the one defining the lines of a pentagram. A line of a pentagram is labelled by a pairwise commuting set of four observables with their product producing the trivial observable III (up to a sign). We have 30 such lines forming the 12 pentagrams of our double-six. It is easy to check that in terms of four weight vectors this condition translates to

$$\Lambda^{(\mathcal{A}_1)} + \Lambda^{(\mathcal{A}_2)} + \Lambda^{(\mathcal{A}_3)} + \Lambda^{(\mathcal{A}_4)} = 0, \quad |\mathcal{A}_s \cap \mathcal{A}_t| = 1, \quad s, t = 1, 2, 3, 4 \quad (68)$$

meaning that the sum of four incident weight vectors should produce the zero vector in \mathbb{R}^6 . In terms of three element subsets this condition means that $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$ should

produce the set with *double occurrence* for all six labels. Two characteristic examples of that kind are the ones

$$\begin{aligned} (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) &= (\{\bar{1}2\bar{3}\}, \{\bar{1}23\}, \{\bar{2}31\}, \{\bar{3}12\}) \\ (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) &= (\{\bar{1}23\}, \{\bar{2}31\}, \{\bar{1}3\bar{1}\}, \{\bar{2}2\bar{3}\}). \end{aligned} \quad (69)$$

It is easy to check that in the first case we have 12 and in the second 18 possibilities yielding the correct number 30.

In order to complete the correspondence one can present the weight diagram labelled by observables of our double-six. Using the dictionary of (64) and Figure 8 for the weight diagram one immediately infers that the simple roots are represented by the observables

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \leftrightarrow (IXI, IZI, XXY, ZII, XII). \quad (70)$$

These observables can be used to label the Dynkin diagram of A_5 , see Figure 9.

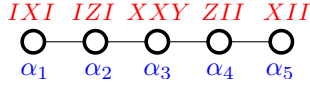


Figure 9: Dynkin diagram of A_5 labelled with observables

As one can check the nodes of the Dynkin diagram are connected if the corresponding observables are not commuting and not connected if the ones are commuting. To the usual rule of subtracting a certain simple root from a weight, corresponds the rule of applying the transvection T_{α_i} to the observable \mathcal{O}_A representing the weight $\Lambda^{(A)}$. If $[\alpha_i, \mathcal{O}_A] = 0$ then \mathcal{O}_A is left invariant however, if $\{\alpha_i, \mathcal{O}_A\} = 0$ for some i then \mathcal{O}_A is transformed to a new observable via multiplying it with the observable which represents α_i . For example

$$T_{\alpha_3} \Lambda^{(123)} = \Lambda^{(\bar{1}23)} \leftrightarrow T_{XXY} IIX = XXZ \quad (71)$$

due to the fact that $\{XXY, IIX\} = 0$ meaning that the corresponding observables are not commuting.

Now as a last step one can display the weight diagram of the 20 dimensional irrep of $SU(6)$ as labelled by the 20 observables of our canonical double-six. In order to present this all we have to do is to transform the labels of the Dynkin diagram of Figure 9 with the involution $T_{ZZX} T_{XXX}$. With this new labelling of the α_i we have new transvections T_{α_i} acting on the new weights this time represented by the preimage of the $1 + 9 + 9 + 1$ observables showing up in the right hand side of (45)-(46). The resulting structure can be seen in Figure 10. As we have stressed one can regard this as the canonically labelled weight diagram related to or canonical double-six. From this canonical labelling any of the 1008 labellings related to the full set of double-sixes of pentagrams can be obtained. The only thing we have to do is to apply our familiar method already has been illustrated by our considerations yielding (45)-(47).

We have clarified the representation theoretic meaning of our double-six of pentagrams via a rather ad hoc analogy to the structure of a special Freudenthal system. This approach was based on the specific Veldkamp line $(H_{YYI}, H_{YYI}, C_{IYY})$ and provided an explicit mapping between weights and observables as shown in (64). In order to get to the same mapping for the *canonical* Veldkamp line $(H_{III}, H_{YYY}, C_{YYY})$ we had to transform back with the involution $T_{ZZX} T_{XXX}$. Can we obtain a more direct approach for the understanding of the mapping of (65) which also gives additional insight into the structure of our class of Veldkamp lines?

The answer to this question is yes. In order to show this let us consider the following set of generators for a seven dimensional Clifford algebra

$$(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7) = (ZYI, YIX, XYI, IXY, YIZ, IZY, YYY) \quad (72)$$

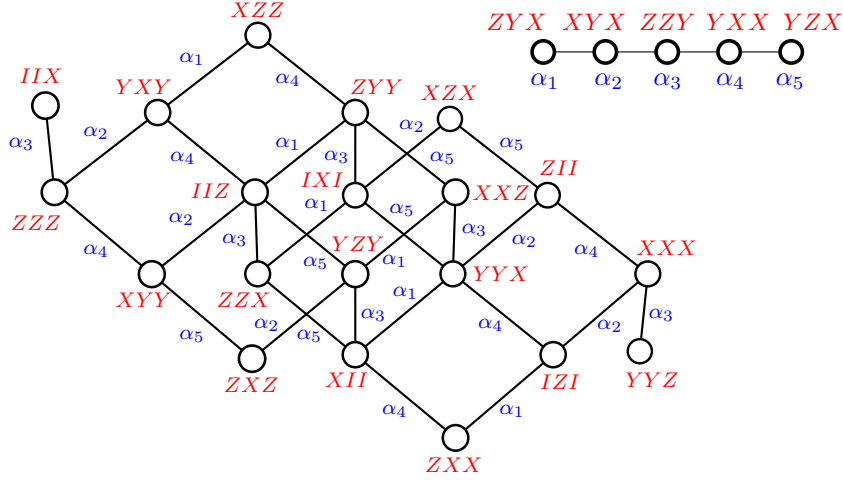


Figure 10: The canonically labelled weight diagram of the 20 dimensional irrep of A_5

satisfying

$$\{\Gamma_I, \Gamma_J\} = 2\delta_{IJ}, \quad I, J = 1, 2, \dots, 7 \quad (73)$$

and

$$i\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7 = III. \quad (74)$$

Let us then consider the following three sets of operators

$$\Gamma_I, \quad \Gamma_I\Gamma_J, \quad \Gamma_I\Gamma_J\Gamma_K, \quad 1 \leq I < J < K \leq 7. \quad (75)$$

It is easy to check that the first two sets contain $7 + 21 = 28$ antisymmetric operators and the third set contains 35 symmetric ones. Consider now the relations above modulo elements of μ_4 . Then the four triangles corresponding to the four subsets of Figure 7 giving rise to the explicit list of operators of (41)-(44) can be labelled as

$$\{\Gamma_\mu\Gamma_\nu\Gamma_\rho\}, \quad \{\Gamma_\mu, \Gamma_{\mu 7}\}, \quad \{\Gamma_\mu\Gamma_\nu, \Gamma_7\}, \quad \{\Gamma_\mu\Gamma_\nu\Gamma_7\} \quad 1 \leq \mu < \nu < \rho \leq 6. \quad (76)$$

One can check that the particular choice of (72) automatically reproduces the set $\{\Gamma_\mu\Gamma_\nu\Gamma_\rho\}$ with our labelling of the 20 operators of the canonical set in terms of the 3 element subsets of the set $\{1, 2, 3, 4, 5, 6\} = \{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$. Indeed, we have

$$\Gamma^{(123)} \leftrightarrow IIX, \quad \begin{pmatrix} \Gamma^{(\bar{1}23)} & \Gamma^{(\bar{1}31)} & \Gamma^{(\bar{1}12)} \\ \Gamma^{(\bar{2}23)} & \Gamma^{(\bar{2}31)} & \Gamma^{(\bar{2}12)} \\ \Gamma^{(\bar{3}23)} & \Gamma^{(\bar{3}31)} & \Gamma^{(\bar{3}12)} \end{pmatrix} \leftrightarrow \begin{pmatrix} ZZZ & YXY & XZZ \\ XYY & IIZ & ZYY \\ ZXZ & YZY & XXZ \end{pmatrix} \quad (77)$$

where we employed the notation $\Gamma^{(\mu\nu\rho)} \equiv \Gamma_\mu\Gamma_\nu\Gamma_\rho$, and as usual the missing entries are obtained by the action of the transvection $T_{YYY} = T_{\Gamma_7}$. Note that by virtue of (74) this transvection indeed acts as the overline involution since for example

$$\Gamma_7\Gamma^{(\bar{1}23)} = \Gamma_7\Gamma_2\Gamma_3\Gamma_4 \simeq \Gamma_1\Gamma_5\Gamma_6 = \Gamma^{(1\bar{2}\bar{3})} \quad (78)$$

where \simeq means equality modulo an element of μ_4 . One can also immediately verify that the transvection T_{Γ_7} exchanges the two components of the set $\{\Gamma_\mu, \Gamma_{\mu 7}\}$ (Schläfli's double-six) and the 15 operators provide a labelling for the doily of Figure 5.

One can also realize that the special structure of our canonical Veldkamp line is related to our special realization of our Clifford algebra. Indeed, all of the operators of (72) are *antisymmetric* ones. However, since all what is important for us is merely commutation properties, we could have used *any* such algebra for producing any other Veldkamp line from our class of 1008 elements. Hence our labelling of (76) encapsulates the general structure of our particular class of Veldkamp lines. The structure of incidence structures based on Clifford algebras has already been studied[38, 39]. In particular it is known that the number of such Clifford algebras (modulo permutations) is 288 and this number splits as $288 = 280 + 8$ where 8 is the number of possibilities for forming a seven dimensional Clifford algebra featuring only antisymmetric operators. The set with 280 elements is formed by Clifford algebras featuring four symmetric and three antisymmetric operators. An example of this kind is obtained by transforming our Clifford algebra of (72) by the transvection $T_{ZZX}T_{XXX}$

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7) = (IXX, IYX, IZX, XIZ, YIZ, ZIZ, IYY). \quad (79)$$

Of course the operators γ_I that can be used to label the operators of the Veldkamp line $(H_{YYI}, H_{YYY}, C_{YYI})$ give rise to the same patterns as the ones showing up in (76).

7 Conclusions

In this paper we investigated the geometry of the space of Mermin pentagrams, objects that are well-known for ruling out noncontextual hidden variable theories as alternatives to quantum theory. It is shown that this space of 12 096 possible pentagrams is organized into 1008 families, with each family containing a "double-six" of pentagrams. We have established an important connection between the structure of the set of these families of Mermin *pentagrams* and a particular class of *Veldkamp lines* for three qubits. This study provided considerable amount of new geometric data on the structure of pentagrams. More importantly for the first time our result established a clear physical meaning of the abstract mathematical notion of a Veldkamp line for three-qubits featuring elliptic and hyperbolic quadrics. In short: a Veldkamp line of that kind encodes information on the geometry of special arrangements of three-qubit observables, in particular of all possible Mermin pentagrams that can be built from three-qubit observables.

The basic building blocks of the space of Mermin pentagrams are double-sixes. We have shown that they are inherently connected to the structure of the weight diagram for the 20 dimensional irrep of $SU(6)$. Double-sixes are comprising a substructure within each Veldkamp line featuring 20 observables. We demonstrated that due to a transitive action of the symplectic group on the 1008 element set of Veldkamp lines it is enough to study merely one of these double-sixes (a one we called *canonical*), the rest of them can be generated in a straightforward manner.

There are many interesting ideas that follow from our considerations. Let us mention just two, the first of them is related to physics and the second to mathematics.

The first of them is the fact that our approach relates the space of Mermin pentagrams to a subset of the space of Mermin squares both built from three-qubit observables. Indeed, let us consider the 9 observables on the right hand side of (45) taken together with the observable IIX . This observable is commuting with all of the 9 observables. These 10 observables form a characteristic substructure from the $1 + 9 + 9 + 1$ structure we have studied. Moreover, IIX taken together with any three observables taken from any of the three diagonals or anti-diagonals of the matrix from the right hand side of (46) forms six sets of quadruplets with pairwise commuting entries. They are six copies from the 30 possible pentagram lines taken from a double-six of pentagrams. Multiplying the observables on the

right hand side of (45) with IIX and rearranging the observables as follows one gets

$$\begin{pmatrix} XXI & XYI & XZI \\ YXI & YYI & YZI \\ ZXI & ZYI & ZZI \end{pmatrix} \mapsto \begin{pmatrix} XXI & YYI & ZZI \\ YZI & ZXI & XYI \\ ZYI & XZI & YXI \end{pmatrix}. \quad (80)$$

Let us call for the arrangement on the right hand side the rows and columns of the matrix as the *lines* of a grid with the operators being its *points*. Then one can check that the grid labelled by three-qubit observables has the properties (P1)-(P3) of Section 3 provided this time (P1) means: three pairwise commuting operators that multiply to $\pm III$. The object we have obtained is called: a Mermin-square. One can transform this object back by the transvection $T_{ZZX}T_{XXX}$ with the result

$$\begin{pmatrix} XXI & YYI & ZZI \\ XIX & XZI & IZX \\ IXX & ZXI & ZIX \end{pmatrix} \quad (81)$$

being another Mermin-square. In this way for each Veldkamp line one obtains a Mermin-square 1008 in total. However, in our construction the observable IIX (the highest weight) was special. But nothing prevents us from repeating the same construction by elevating *any* of the 20 observables to the status of a highest weight. Hence the total number of Mermin squares obtained in this way seems to be 20 160. However, since the highest and lowest weight yields the same Mermin-square the correct number is 10 080. Interestingly this number is just the half of the total number of Mermin-squares that can be built from three-qubit observables[40].

One can understand this result in yet another way. Let us recall the relevant results of Section 4, in particular (38)-(39). Mermin pentagrams per Veldkamp line are built from the 20 observables forming 30 lines comprising *quadruplets* of observables residing in the double-six part of the Veldkamp line. Mermin-squares per Veldkamp line on the other hand are built from the 15 observables forming the 15 lines comprising *triples* of observables residing in the doily (the core set) part of the Veldkamp line. It is well-known[24] that a particularly labelled doily is containing 10 Mermin-squares (grids) as geometric hyperplanes. Since we have 1008 such doilies as core sets the naive count gives 10 080 Mermin squares. The precise nature of this Mermin-square vs. Mermin-pentagram correspondence will be addressed in future work.

The second of the ideas that follows from our work is representation theoretical. From our investigations it is clear that the notion of a particular Veldkamp line for three-qubits contains valuable representation theoretic information in a finite geometric manner. Apart from the structure of the double-six of pentagrams part which encodes the weight diagram of the 20 of A_5 , it is clear that the Klein-quadric (ie. the H_{III}) part featuring 35 observables should be related to the 35 of A_6 . Indeed, due to $\binom{7}{3} = 35$ the relevant representation in this case is the one on trivectors in a *seven* dimensional space. On the other hand, as already mentioned in the Introduction the elliptic quadric part of our canonical Veldkamp line (i.e. the H_{YYY} part) is encoding information on the weight diagram of the 27 of E_6 . Hence by a convenient labelling for the E_6 Dynkin diagram one should be able to reproduce a labelling of the weight diagram of the 27 dimensional irrep in a manner similar to Figure 9.

This connection between representation theory and finite geometry should of course survive for other Veldkamp lines and other multiqubit observables. In particular we conjecture that the $N = 2, 3, 4$ cases should produce some of the low dimensional representations of the simply laced groups (the ones of ADE-type) in a finite geometric setting. We are planning to explore these fascinating ideas in a future work.

Finally, let us note that our findings can also pave the way for further studies for uncovering the connection found between noncontextual configurations and weight diagrams

of certain representations of Lie-groups. We even conjecture that by studying other Veldkamp lines of the Veldkamp space for N -qubits makes it possible to find new contextual configurations. This observation might be a starting point to study a certain subset of such configurations in a systematic manner. Our mathematical framework based on the notion of Veldkamp space encoding information on contextual configurations via representation theory might establish a unified picture. Our hope is that our Veldkamp space based approach might even serve as a guiding principle for organizing some of the scattered results relating group representations with contextuality.

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References

- [1] E. P. Specker, *Dialectica* **14**, 239 (1960); S. Kochen, E. P. Specker, *J. Math. Mech.* **17**, 59 (1967).
- [2] J. S. Bell, *Rev. Mod. Phys.* **38**, 447 (1966); Reprinted in J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press, Cambridge, UK, 1987.
- [3] A. Peres, *J. Phys. A* **24**, L175 (1991).
- [4] A. Peres, *Quantum Theory: Concepts and Methods*, Kluwer Academic Publishers, (2002).
- [5] N. D. Mermin, *Phys. Rev. Lett.* **65**, 3373 (1990).
- [6] N. D. Mermin, *Rev. Mod. Phys.* **65**, 803 (1993).
- [7] D. M. Greenberger, M. A. Horne and A. Zeilinger in *Bell's Theorem, Quantum Theory and Conceptions of the universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989); D. M. Greenberger, M. A. Horne, A. Shimony and A. Zeilinger: *Am. J. Phys.* **58**, 1131 (1990).
- [8] M. Waegell and P. K. Aravind, *Phys. Lett. A* **377**, 546 (2013); *J. Phys. A: Math. Theor.* **45**, 405301 (2012); *Phys. Rev. A* **88**, 012102 (2013).
- [9] M. Waegell and P. K. Aravind, *J. Phys. A: Math. Theor.* **45** 405301 (2012). <http://arxiv.org/abs/1205.5015>
- [10] M. Planat, M. Saniga and F. Holweck, *Quantum Information Processing* **12**, 2535 (2013). <https://arxiv.org/abs/1212.2729>
- [11] P. Lévay, M. Planat and M. Saniga, *Journal of High Energy Physics* **9**, 037 (2013). <http://arxiv.org/abs/1305.5689>
- [12] M. Planat, *Information* **5** 209 (2014).
- [13] M. Saniga and P. Lévay, *Europhysics Letters* **97** 50005 (2012).
- [14] P. Lévay, M. Saniga, P. Vrana, *Phys. Rev. D* **78** 124022, (2008). <http://arxiv.org/abs/0808.3849>
- [15] P. Lévay, M. Saniga, P. Vrana, P. Pracna, *Phys. Rev. D* **79** 084036, (2009). <http://arxiv.org/abs/0903.0541>

- [16] L. Borsten, M. J. Duff and P. Lévy, Classical and Quantum Gravity 29 (22), 224008 (2012).
- [17] M. Saniga, M. Planat, Quantum Information and Computation **8**, 127 (2008), Advanced Studies in Theoretical Physics **1**, 1 (2007).
- [18] Buekenhout F., Cohen A.M., Diagram geometry, Springer (2009)
- [19] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press 2000.
- [20] D. Gottesman, Phys. Rev. **A54**, 1862 (1996), D. Gottesman, Phys. Rev. **A57** 127 (1998).
- [21] A. R. Calderbank, E. M. Rains, P. W. Shor and N. J. A. Sloane, Phys. Rev. Lett. **78** 405 (1997).
- [22] E. Shult, Bull. Belg. Math. Soc. **4**, 299 (1997).
- [23] E. Shult, Points and Lines: Characterizing the Classical Geometries, Chapter 4.1., Springer-Verlag Berlin Heidelberg (2011).
- [24] M. Saniga and M. Planat, P. Pracna, SIGMA **3**, 75 (2007). <http://arxiv.org/abs/0704.0495v3>
- [25] P. Vrana, P. Lévy, J. PhysA: Math. Theor. **A43**, 125303 (2010). <http://arxiv.org/abs/0906.3655v3>.
- [26] N. Bourbaki, *Elements of Mathematics, Lie Groups and Lie Algebras*, Chapters 4-6, Masson, Springer-Verlag Berlin Heidelberg (2002), see on page 243.
- [27] O. T. O'Meara, *Symplectic groups*, Chapter 2.1, American Mathematical Society, USA (1978) Yaim Cooper. *Generators of the Symplectic Group*. 2005. <http://www-math.mit.edu/~dav/sympgen.pdf>.
- [28] B. L. Cherchiai, B. van Geemen, J. Math. Phys. **51**, 122203 (2010), <http://arxiv.org/abs/1003.4255>
- [29] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, Pitman Boston-London-Melbourne (1984).
- [30] M. J. Duff, S. Ferrara, Phys. Rev. D**76** 124023 (2007).
- [31] F. Holweck and M. Saniga, *Contextuality with a Small number of Observables*, <http://arXiv.org/abs/1607.07567v1>
- [32] Zsolt Szabó, *The finite Geometric Aspects of Quantum Contextuality*, BSc Thesis (in Hungarian), Institute of Physics, Budapest University of Technology and Economics (2016).
- [33] B. Polster, *A Geometric Picture Book*, Springer-Verlag New York (1998).
- [34] P. Lévy and P. Vrana, Phys. Rev. A**78**, 022329 (2008). <http://arxiv.org/abs/0806.4076v1>
- [35] P. Vrana and P. Lévy, J. Phys. A: Math. Theor. **42**, 285303 (2009). <http://arxiv.org/abs/0902.2269v2>
- [36] S. Krutelevich, Journal of Algebra, **314**, 924 (2007).

- [37] R. Slansky, Physics Reports **79**, 1-128 (1981).
- [38] R. Shaw, *Finite geometry, Dirac groups and the table of real Clifford algebras* pp. 59-99, in R. Ablamowicz and P. Lounesto, eds., Clifford Algebras and Spinor Structures (Kluwer Acad. Pubs., Dordrecht, 1995).
- [39] R. Shaw, J. Phys. A: Math. Gen. **21** 7-16 (1988).
- [40] M. Planat, private communication (2016).